

Covariant jump conditions in electromagnetism

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Abstract

A generally covariant four-dimensional representation of Maxwell's electrodynamics in a generic material medium can be achieved straightforwardly in the metric-free formulation of electromagnetism. In this setup, the electromagnetic phenomena described by two tensor fields, which satisfy Maxwell's equations. A generic tensorial constitutive relation between these fields is an independent ingredient of the theory. By use of different constitutive relations (local and non-local, linear and non-linear, etc.), a wide area of applications can be covered. In the current paper, we present the jump conditions for the fields and for the energy-momentum tensor on an arbitrarily moving surface between two media. From the differential and integral Maxwell equations, we derive the covariant boundary conditions, which are independent of any metric and connection. These conditions include the covariantly defined surface current and are applicable to an arbitrarily moving smooth curved boundary surface. As an application of the presented jump formulas, we derive a Lorentzian type metric as a condition for existence the wave front in isotropic media. This result holds for the ordinary materials as well as for the metamaterials with the negative material constants.

Keywords: Metric-free electrodynamics; relativity; jump conditions.

1 Introduction

Maxwell's system of the electromagnetic field equations was initially formulated as a collection of the experimentally established electromagnetic laws, i.e., as a phenomenological model. Eventually, this model was recognized as a Lorentz covariant and even general relativistic covariant theory, see [1],[2],[3],[4]. The rich group of transformations associated with the Maxwell system is not only an aesthetic feature. In fact, the general covariant form of the equations is very useful even on a flat spacetime when curvilinear coordinates are used. It becomes compulsory when the contribution of the gravitational field to electromagnetism is considered. Moreover, the generally covariant form can serve as a guiding

framework for possible theoretical extensions of the classical electromagnetic theory.

The procedure of the reformulation of the Maxwell system into a covariant form requires a proper tensorial redefinition of the basic electromagnetic variables. It is quite well known from Lorentz, Minkowski, and Einstein. A historical account of this development was given recently in [8] and [9].

Another principal feature of Maxwell's system is that it is not dependent from the geometry of the underlying manifold, neither from the metric nor from the connection. This fact was recognized long ago [10],[11],[12],[13], but its comprehensive treating was provided only recently, see [5] and the references given therein. In vacuum, the metric-independence of the Maxwell system is just a nice mathematical property, but in electromagnetism of media this fact has a firm physical meaning. Indeed, the only way one can observe the spacetime metric in a non-trivial electromagnetic medium is by studying the propagation of light in it. However, the light propagation is governed by the electromagnetic moduli of the medium, not by the Lorentzian metric. In general, the corresponding optical geometry is not Lorentzian. Instead, it obeys a Finslerian geometry [14]. Thus, the Lorentzian metric tensor of the vacuum geometry cannot actually be observed inside of electromagnetic media. Under these circumstances, the metric tensor turns out to be a redundant notion that must be omitted from our description. In fact, the metric-free formulation is acceptable in a vast range of the pure electromagnetic problems, especially in optics. Actually, when a proper definition of the variables is used, the metric tensor "cancels out" from all equations. Some examples of such a metric-free representation were derived recently: (i) The generalized covariant dispersion relation [5], [15]. (ii) The generalized covariant photon propagator [16]. (iii) The non-relativistic dispersion relation for anisotropic media [17]. Particularly, the axion, skewon and dilaton partners of the photon naturally emerge in this framework. The generic metric-free construction also can serve as a working model to organize the experimental electromagnetic data in order to estimate the possible deviations from the standard Maxwell theory [6], [7].

In the current paper, we turn to an additional ingredient of the Maxwell theory — *the jump conditions* for the fields and for the energy-momentum tensor on an arbitrarily moving surface between two different media. These relations are usually presented in a non-relativistic three-dimensional form with an explicit use of the Euclidean metric. Our aim is to derive the metric-free and general relativistic covariant form of the jump conditions. This question was studied rather intensively, see [18], [19], [20], [21], [22], [23]. However, to our knowledge, a manifestly covariant and metric-free result was not yet presented in the literature.

The organization of the paper is as follows: In the next section, we recall the covariant metric-free formulation of the Maxwell system. The third section is devoted to a derivation of the covariant jump conditions for electromagnetic field on an arbitrary moving surface. We start with the known static jump conditions and show how they can be rewritten in the metric-free form. A complete description of the field jumps must include surface electric charges and currents.

The covariant metric-free description of the surface quantities dictates the use of the Dirac δ -form instead of the Dirac δ -function. The covariant meaning of the boundary hypersurface in the four-dimensional manifold is discussed and the covariant metric-free definition of the surface electric current introduced. First, we derive the metric-free covariant jump conditions from the Maxwell equations by formal differentiation of the Heaviside step-distribution. In parallel with that, we present a derivation of the same jump conditions from the integral Maxwell equations. We study the $(1+3)$ -decomposition of the covariant jump conditions. In the static case, the classical jump conditions are reinstated. We also discuss the number of the independent boundary conditions. Their number turns out to be six, both in the covariant description and in the $(1+3)$ decomposition. An example in section 8 represents the use of the covariant metric-free jump conditions in the case of a uniform isotropic media. The consistence of the jump conditions induce the possibility for the existence of shock waves in this medium. Their front has a Lorentzian-type gradient with a speed of $v = 1/\sqrt{\epsilon\mu}$ instead of the vacuum speed of light. Thus, the Lorentzian-type metric is obtained in the initially metric-free formulation.

We complete the paper with a discussion of the results.

2 Maxwell equations

In this section, we give a brief account of the covariant metric-free presentation of the electromagnetism in media. For a comprehensive description, see [5] and the references given therein. Let us start with a differential forms formulation of the Maxwell equations. On an arbitrary 4-dimensional differential manifold, the two field equations read

$$dF = 0, \quad d\mathcal{H} = \mathcal{J}. \quad (2.1)$$

Here F is an untwisted 2-form of the *field strength*, \mathcal{H} is a twisted 2-form of the *field excitation*, and \mathcal{J} is a twisted 3-form of the *electric current*. In an integral formulation, the equations (2.1) are represented by a system

$$\int_{\partial\Sigma} F = 0, \quad \int_{\partial\Sigma} \mathcal{H} = \int_{\Sigma} \mathcal{J}, \quad (2.2)$$

where Σ is an arbitrary connected 3-dimensional domain bounded by the closed 2-dimensional surface $\partial\Sigma$. In fact, one can interpret these equations as the expressions of the magnetic flux conservation law and the electric charge conservation law, respectively.

The systems (2.1) and (5.1) are explicitly general relativistic covariant and do not require a metric for their formulation. In a coordinate basis, the differential forms can be expanded as

$$F = \frac{1}{2} F_{ij} dx^i \wedge dx^j, \quad \mathcal{H} = \frac{1}{2} \mathcal{H}^{ij} \epsilon_{ijkl} dx^k \wedge dx^l. \quad (2.3)$$

Here, the spacetime coordinates $\{x^0, x^1, x^2, x^3\} = \{t, x, y, z\}$ are used. The Roman indices run over the range of $i, j, \dots = 0, 1, 2, 3$. We use the 4-dimensional Levi-Civita permutation pseudo-tensor ϵ^{ijkl} which is normalized as $\epsilon^{0123} = 1$. The odd 3-form of the electric current is expanded as

$$\mathcal{J} = \frac{1}{6} \mathcal{J}^i \epsilon_{ijkl} dx^j \wedge dx^k \wedge dx^l. \quad (2.4)$$

With the definitions above the tensorial form of Maxwell's equations is given by

$$\epsilon^{ijkl} F_{jk,l} = 0, \quad \mathcal{H}^{ij}_{,i} = \mathcal{J}^j. \quad (2.5)$$

The $(1+3)$ -decomposition of the field tensors F_{ij} and \mathcal{H}^{ij} reads

$$F_{0\alpha} = E_\alpha, \quad F_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma} B^\gamma, \quad (2.6)$$

$$\mathcal{H}^{0\alpha} = -D^\alpha, \quad \mathcal{H}^{\alpha\beta} = \epsilon^{\alpha\beta\gamma} H_\gamma, \quad (2.7)$$

where the Greek indices run over the range $\alpha, \beta = 1, 2, 3$. The inverse relations are given by

$$E_\alpha = F_{0\alpha}, \quad B^\alpha = -\frac{1}{2} \epsilon^{\alpha\beta\gamma} F_{\beta\gamma}, \quad (2.8)$$

$$D^\alpha = -\mathcal{H}^{0\alpha}, \quad H_\alpha = \frac{1}{2} \epsilon_{\alpha\beta\gamma} \mathcal{H}^{\beta\gamma}. \quad (2.9)$$

The correspondence between these two sets of formulas can be checked by use of the relations $\epsilon_{\alpha\beta\gamma} \epsilon^{\alpha\mu\nu} = \delta_\beta^\mu \delta_\gamma^\nu - \delta_\beta^\nu \delta_\gamma^\mu$ and $\epsilon_{\alpha\beta\gamma} \epsilon^{\alpha\beta\mu} = 2\delta_\gamma^\mu$. The $(1+3)$ -decomposition of the electric current J^i is given by

$$\mathcal{J}^0 = \rho, \quad \mathcal{J}^\alpha = j^\alpha. \quad (2.10)$$

When these definitions are substituted into (2.5), we come to the ordinary 3D-representation of Maxwell's equations in media, see [4],

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad (2.11)$$

$$\nabla \cdot \mathbf{D} = \rho, \quad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{j}. \quad (2.12)$$

The definitions for the vector operations are assumed here as

$$\nabla \cdot \mathbf{B} = B^\alpha_{,\alpha}, \quad \nabla \cdot \mathbf{D} = D^\alpha_{,\alpha}, \quad (2.13)$$

and

$$\nabla \times \mathbf{H} = -\epsilon^{\alpha\beta\gamma} H_{\beta,\gamma}, \quad \nabla \times \mathbf{E} = -\epsilon^{\alpha\beta\gamma} E_{\beta,\gamma}, \quad (2.14)$$

where $\epsilon^{\alpha\beta\gamma}$ denotes the three-dimensional Levi-Civita's permutation pseudo-tensor with $\epsilon^{123} = 1$.

It is an extremely significant fact that the 3-dimensional system (2.11-2.12) is already general relativistic invariant and metric-free. Indeed, even being written in the usual three-dimensional notations it is equivalent to the invariant system (2.1) and independent of a metric structure.

3 Jump conditions

In this section, we start with the standard textbook boundary conditions, which are the consequences of Maxwell's field equations. These conditions include 3-component vector fields depending of time and position. Certainly, they are not preserved under 4-dimensional transformations. Moreover, the conditions deal with a static flat boundary which is also a non-covariant notion. Furthermore, these expressions use explicitly the Euclidean metric of the three-dimensional position space. Observe that the underlying geometric structure is ill defined. Due to the independent free variables, the space must be 4-dimensional. The fields, however, are assumed to be described by 3-vectors.

Our aim is to remove all these restrictions in order to obtain the covariant metric-free 4-dimensional boundary conditions.

3.1 Static jump conditions

Let us first consider a non-moving surface between two media. In a chosen system of Cartesian spatial coordinates $\{x, y, z\}$, a static surface can be represented by an implicit equation

$$\varphi(x, y, z) = 0 \quad - \quad \text{surface.} \quad (3.1)$$

We assume $\varphi(x, y, z)$ to be a smooth real function defined in the whole space, i.e., $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}$. For an arbitrary point (x, y, z) in the medium, we assume one of the conditions

$$\varphi(x, y, z) < 0 \quad - \quad \text{medium (1),} \quad (3.2)$$

or,

$$\varphi(x, y, z) > 0 \quad - \quad \text{medium (2).} \quad (3.3)$$

For an arbitrary physical variable $G = G(t, x, y, z)$, we describe its discontinuity at the boundary $\varphi(x^\alpha)$ by the quantity

$$[G] = \lim_{\varphi \rightarrow 0^+} G(t, x, y, z) - \lim_{\varphi \rightarrow 0^-} G(t, x, y, z). \quad (3.4)$$

For a static boundary surface, one introduces a normal unit vector

$$\mathbf{n} = \frac{\nabla \varphi}{\|\nabla \varphi\|} \quad (3.5)$$

and writes the standard static 3-dimensional boundary conditions as [4]

$$\mathbf{n} \cdot [\mathbf{B}] = 0, \quad \mathbf{n} \times [\mathbf{E}] = 0, \quad (3.6)$$

$$\mathbf{n} \cdot [\mathbf{D}] = \sigma, \quad \mathbf{n} \times [\mathbf{H}] = \mathbf{K}. \quad (3.7)$$

Here σ and \mathbf{K} are the 3-dimensional surface charge scalar and surface current vector respectively. At first glance, the conditions (3.6–3.7) depend crucially on

the Euclidean space metric. Indeed, the 3-dimensional metric is involved in the definition of the unit normal \mathbf{n} and of the scalar (dot) product. We will see that these equations can be readily reformulated in a metric-free form.

First, we substitute (3.5) into (3.6-3.7) to obtain

$$\nabla\varphi \cdot [\mathbf{B}] = 0, \quad \nabla\varphi \times [\mathbf{E}] = 0, \quad (3.8)$$

$$\nabla\varphi \cdot [\mathbf{D}] = \sigma \|\nabla\varphi\|, \quad \nabla\varphi \times [\mathbf{H}] = \mathbf{K} \|\nabla\varphi\|. \quad (3.9)$$

The scalar product terms appearing in these equations can be written as a contraction of a covector $\nabla\varphi$ with the vector fields $\mathbf{B} = B^\alpha \partial_\alpha$ and $\mathbf{D} = D^\alpha \partial_\alpha$. Note that such a description of the fields \mathbf{B} and \mathbf{D} as contravariant vectors is necessary for the covariant meaning of the field equations. Alternatively, $\nabla\varphi$ comes as a covariant vector (or covector). Consequently, we have the identifications

$$\nabla\varphi \cdot \mathbf{B} = \varphi_{,\alpha} B^\alpha, \quad \nabla\varphi \cdot \mathbf{D} = \varphi_{,\alpha} D^\alpha. \quad (3.10)$$

The standard expression for the cross-product also can be assembled in a non-metrical form if the corresponding fields are treated as down-indexed, i.e., as covectors. As we have already seen, this is the case with the fields \mathbf{E} and \mathbf{H} . Thus, we have a metric-free representation

$$(\nabla\varphi \times \mathbf{E})^\alpha = \varepsilon^{\alpha\beta\gamma} \varphi_{,\beta} E_\gamma, \quad (\nabla\varphi \times \mathbf{H})^\alpha = \varepsilon^{\alpha\beta\gamma} \varphi_{,\beta} H_\gamma. \quad (3.11)$$

Consequently, the system (3.6-3.7) takes the form

$$\varphi_{,\alpha} [B^\alpha] = 0, \quad \varepsilon^{\alpha\beta\gamma} \varphi_{,\beta} [E_\gamma] = 0, \quad (3.12)$$

$$\varphi_{,\alpha} [D^\alpha] = \sigma \|\nabla\varphi\|, \quad \varepsilon^{\alpha\beta\gamma} \varphi_{,\beta} [H_\gamma] = \mathcal{K}^\alpha \|\nabla\varphi\|. \quad (3.13)$$

Substituting (2.8), we can rewrite this system via the components of the 4-dimensional tensors as

$$\varepsilon^{\alpha\beta\gamma} \varphi_\alpha [F_{\beta\gamma}] = 0, \quad \varepsilon^{\alpha\beta\gamma} \varphi_\beta [F_{0\gamma}] = 0, \quad (3.14)$$

$$\varphi_\alpha [\mathcal{H}^{\alpha 0}] = \sigma \|\nabla\varphi\|, \quad \varphi_\beta [\mathcal{H}^{\alpha\beta}] = \mathcal{K}^\alpha \|\nabla\varphi\|. \quad (3.15)$$

Now the metric tensor remains only in the definition of the norm $\|\nabla\varphi\|$. It will be removed in the sequel by a suitable redefinition of the surface quantities. Note that the system of these equations is still 3-dimensional and the boundary between two media is still static.

3.2 Moving boundary

A covariant definition of a boundary between two media by necessity must be time-dependent. On a four-dimensional manifold, the boundary spans a three-dimensional hypersurface. The general representation of such a hypersurface is given by a scalar equation

$$\varphi(x^i) := \varphi(t, x, y, z) = 0 \quad - \quad \text{surface.} \quad (3.16)$$

This equation can describe a submanifold of a rather complicated topology, for instance, with self-intersections. In order to remove these cases, it is enough to require the gradient of φ to be non-zero at all points where (3.16) holds.

Moreover, different functional dependence of φ on the time coordinate can be used to describe the same boundary. It should be noted that, for a given hypersurface, a choice of a function $\varphi(x^i)$ is highly non-unique. Indeed, for an arbitrary monotonic function f with $f(0) = 0$, two equations $f(\varphi) = 0$ and $\varphi = 0$ represent the same hypersurface. Moreover, the physical dimensions of the quantities φ and $f(\varphi)$ can be different. We will take care of these difficulties subsequently.

Not all of the surfaces (3.16) can serve as a physical boundary between two media. We restrict ourself to a small piece of the surface which is embedded without self-intersections in that part of the $4D$ manifold where the coordinates x^i are defined. Global embedding problems will not play any role in our considerations. We assume that $\varphi(t, x, y, z)$ is a smooth function in the whole domain. Moreover, we restrict our considerations to a small region which is separated by the surface (3.16) into two disjoint pieces representing two media. Consequently, for an arbitrary point into the first medium, we assume

$$\varphi(t, x, y, z) < 0 \quad - \quad \text{medium (1)}, \quad (3.17)$$

while, for a point into the second medium, we have

$$\varphi(t, x, y, z) > 0 \quad - \quad \text{medium (2)}. \quad (3.18)$$

Denote the 4-dimensional gradient of $\varphi(x^i)$ as

$$\varphi_{,i} = \frac{\partial \varphi}{\partial x^i} \quad (3.19)$$

and assume it to be defined and non-zero in the whole spacetime domain. Observe that this quantity naturally comes with a lower index. Hence, it must be treated as a 4-covector. Note that $\varphi_{,i}$ does not depend of the metric tensor and it cannot be treated as a normal to the surface (the orthogonality notion is not defined in the metric-free context). For a static boundary, the identification $\varphi_{,i} = (0, \mathbf{n})$ can be assumed. In the notation of differential forms, we have an even 1-form

$$d\varphi = \varphi_{,i} dx^i, \quad (3.20)$$

whose components are the same as in (3.19).

3.3 Surface current

For a fully covariant boundary condition, we need the notion of a covariant surface current. We treat the electric surface current as a special type of an ordinary electric current which can be substituted into the right-hand side of the inhomogeneous Maxwell equation and if integrated over a 3-dimensional surface, yields the total charge. Consequently, the four-dimensional surface current must

be represented by a differential 3-form with a support on the surface. To deal with such singular currents, which are localized only on a surface $\varphi(x^i) = 0$, we define an even 1-form

$$\mathcal{D}(\varphi) = \delta(\varphi)d\varphi, \quad (3.21)$$

referred to as *Dirac's delta-form* [27]. This is a type of a de Rham current, i.e., a singular differential form [26]. Since the dimension of $\delta(\varphi)$ is inverse to the dimension of φ , the 1-form $\mathcal{D}(\varphi)$ is dimensionless independently from the different possibilities to attach a physical dimension to the function φ . Moreover, the form $\mathcal{D}(\varphi)$ is the same for different functional representations of the hypersurface. Particularly, we can describe the surface by the equation $\varphi(x^i) = 0$ as well as by an infinite family of equations $f(\varphi(x^i)) = 0$, with an arbitrary monotonic function f . For such different functional representations, we will have different delta-functions with the same delta-form (up to the leading sign). Indeed, for a monotonic $\psi = f(\varphi)$,

$$\mathcal{D}(\psi) = \delta(f(\varphi))d(f(\varphi)) = \frac{1}{|f'|}\delta(\varphi)f'd\varphi = \pm\mathcal{D}(\varphi). \quad (3.22)$$

Similarly to the known representation of the Dirac delta-function as a formal derivative of the step-function, the Dirac delta-form can be viewed as the exterior differential of the step-function $u(\varphi)$

$$\mathcal{D}(\varphi) = d(u(\varphi)). \quad (3.23)$$

In a coordinate basis, (3.21) reads

$$\mathcal{D}(\varphi) = \delta(\varphi)\varphi_{,i}dx^i. \quad (3.24)$$

We observe that in tensorial equations (written with explicit indices), the term $\delta(\varphi)\varphi_{,i}$ has an invariant meaning, but not $\delta(\varphi)$ itself. The delta-function $\delta(\varphi)$, when applied to the boundary function $\varphi(x^i)$, is not well-defined and can come with different physical dimensions.

Now we are able to deal with a singular 3-form of an electric surface current in a 4-dimensional space. For a given surface $\varphi(x) = 0$, we define a twisted form of an electric surface current as

$${}^{(sur)}\mathcal{J} = L \wedge \mathcal{D}(\varphi) = L\delta(\varphi) \wedge d\varphi, \quad (3.25)$$

where L is an arbitrary smooth twisted regular 2-form. This definition guarantees explicitly the principal properties of the surface current:

- (i) The surface electric current in the 4D-manifold is given as a 3-form. Hence it is suitable for being substituted into the inhomogeneous field equation (2.1).
- (ii) The current is localized only on the hypersurface $\varphi = 0$. This property is provided by the factor $\delta(\varphi)$. Thus, for $\varphi \neq 0$ (outside the surface), the current vanishes.

- (iii) The current is tangential to the hypersurface. This property is guaranteed by the factor $d(\varphi)$. The tangential relation is represented in a metric-free form as

$${}^{(sur)}\mathcal{J} \wedge d\varphi = 0. \quad (3.26)$$

- (iv) The absolute dimension of the 2-form L and of the 3-form ${}^{(sur)}\mathcal{J}$ are the same, namely the dimension of a charge.
- (v) The surface current is independent of the (uncontrolled) choice of the boundary function $\varphi(x^i)$.

To have a coordinate expression for the surface current (3.25), we recall the usual representation of a 2-form

$$L = \frac{1}{2} L_{ij} dx^i \wedge dx^j. \quad (3.27)$$

When it is combined with (3.24), we have

$${}^{(sur)}\mathcal{J} = \frac{1}{2} L_{ij} \varphi_{,k} \delta(\varphi) dx^i \wedge dx^j \wedge dx^k. \quad (3.28)$$

Comparing with the ordinary representation of the 3-forms (2.10), we obtain

$${}^{(sur)}J_{ijk} = L_{[ij} \varphi_{,k]} \delta(\varphi), \quad (3.29)$$

where the antisymmetrization of the indices is denoted by square parentheses. The corresponding dual tensor $J^i = \varepsilon^{ijkl} J_{jkl}$ is expressed by Levi-Civita dual of the tensor L_{ij}

$${}^{(sur)}J^i = L^{ij} \varphi_{,j} \delta(\varphi), \quad \text{where} \quad L^{ij} = (1/2) \varepsilon^{ijkl} L_{kl}. \quad (3.30)$$

4 Boundary conditions from the differential Maxwell equations

Let us now derive the covariant boundary conditions directly from the Maxwell equations.

4.1 Homogeneous conditions

The discontinuous 2-form F can be written with the use of the Heaviside step-function $u(t)$,

$$u(\varphi) = \begin{cases} 1 & \text{if } \varphi > 0, \\ 0 & \text{if } \varphi < 0. \end{cases}$$

The value $u(0)$ is not prescribed, but assumed to be final. We write the 2-form of the field strength in the whole space as

$$F(x^i) = {}^{(+)}F(x^i)u(\varphi) + {}^{(-)}F(x^i)u(-\varphi) = \begin{cases} {}^{(+)}F(x^i) & \text{if } \varphi(x^i) > 0, \\ {}^{(-)}F(x^i) & \text{if } \varphi(x^i) < 0. \end{cases} \quad (4.1)$$

Calculating the exterior derivative of both sides of (4.1), we have

$$\begin{aligned} dF &= d\left({}^{(+)}F\right)u(\varphi) + d\left({}^{(-)}F\right)u(-\varphi) \\ &\quad + {}^{(+)}F \wedge d\left(u(\varphi)\right) + {}^{(-)}F \wedge d\left(u(-\varphi)\right). \end{aligned} \quad (4.2)$$

The right-hand side of this equation includes terms of two different types — the regular (bounded) terms, given in the first line, and the singular (unbounded) terms, listed in the second line. The left-hand-side of (4.2) is also singular. First, we observe that due to the local equations,

$$d\left({}^{(-)}F\right) = d\left({}^{(+)}F\right) = 0. \quad (4.3)$$

Thus, the regular terms are equal to zero. A precise meaning of the formal differential equations of the singular terms can be given as soon as the integrations are performed on both sides of the equation over some proper regions. Consider a piece Σ of a 3D-surface which is transversal to the surface $\varphi(x^i) = 0$. Let the integral of a singular 3-form dF be *defined* due to Stokes' theorem as

$$\int_{\Sigma} dF := \int_{\partial\Sigma} F. \quad (4.4)$$

Observe that the right-hand-side includes an integration of a non-singular form. Consequently, the left-hand side of this equation is an unambiguous quantity. Now we can describe (4.2) as an equality between the integrands of the well-defined integrals

$$\int_{\Sigma} dF = \int_{\Sigma} {}^{(+)}F \wedge d\left(u(\varphi)\right) + \int_{\Sigma} {}^{(-)}F \wedge d\left(u(-\varphi)\right). \quad (4.5)$$

Let us assume that the homogeneous Maxwell equation, $dF = 0$, holds in the whole space also for the singular forms. Thus, we are left with

$$\int_{\Sigma} {}^{(+)}F \wedge d\left(u(\varphi)\right) + \int_{\Sigma} {}^{(-)}F \wedge d\left(u(-\varphi)\right) = 0. \quad (4.6)$$

Replacing the exterior derivative of the Heaviside step-function by Dirac's delta-form, we have

$$\int_{\Sigma} {}^{(+)}F \wedge \delta(\varphi)d\varphi - \int_{\Sigma} {}^{(-)}F \wedge \delta(-\varphi)d\varphi = 0. \quad (4.7)$$

Since delta-function is even, $\delta(-\varphi) = \delta(\varphi)$, we can rewrite it as

$$\int_{\Sigma} \left[{}^{(-)}F(x^i) - {}^{(+)}F(x^i) \right] \wedge \delta(\varphi)d\varphi = 0. \quad (4.8)$$

Because of the factor $\delta(\varphi)$, we can replace the arguments of the F 's by their limiting values similarly as in (3.4). Consequently,

$$\int_{\Sigma} [F] \wedge \mathcal{D}(\varphi) = 0. \quad (4.9)$$

This equation must hold for an arbitrary surface Σ . Thus, we have the homogeneous boundary condition in a compact covariant metric-free form as

$$[F] \wedge d\varphi = 0. \quad (4.10)$$

Here, the coordinates in $F = F(x^i)$ are assumed to be constrained by $\varphi(x^i) = 0$.

4.2 Inhomogeneous conditions.

Analogously to the field strength F , the excitation field \mathcal{H} in the whole space is written as a combination

$$\mathcal{H} = {}^{(+)}\mathcal{H}u(\varphi) + {}^{(-)}\mathcal{H}u(-\varphi). \quad (4.11)$$

We assume the generic electric current to be a sum of the surface and bulk currents

$$J = {}^{(sur)}J + {}^{(bulk)}J. \quad (4.12)$$

We also assume that the bulk current can be decomposed as

$${}^{(bulk)}J = {}^{(+)}Ju(\varphi) + {}^{(-)}Ju(-\varphi). \quad (4.13)$$

Calculating the exterior derivative of both sides of (4.11), we have

$$\begin{aligned} d\mathcal{H} &= d\left({}^{(+)}H\right)u(\varphi) + d\left({}^{(-)}H\right)u(-\varphi) \\ &\quad + {}^{(+)}H \wedge d\left(u(\varphi)\right) + {}^{(-)}H \wedge d\left(u(-\varphi)\right). \end{aligned} \quad (4.14)$$

Here, the right-hand-side also includes the regular terms, which we list in the first line, and the singular terms listed in the second line. The regular terms are compensated by the bulk current due to the local (point-wise) equations

$$d\left({}^{(\pm)}H\right) = {}^{(\pm)}J. \quad (4.15)$$

Consequently, we are left with the equation

$${}^{(+)}H \wedge d\left(u(\varphi)\right) + {}^{(-)}H \wedge d\left(u(-\varphi)\right) = {}^{(sur)}J. \quad (4.16)$$

The precise meaning of this equation is expressed by the integral equation between the singular terms

$$\int_{\Sigma} {}^{(+)}H \wedge d\left(u(\varphi)\right) + \int_{\Sigma} {}^{(-)}H \wedge d\left(u(-\varphi)\right) = \int_{\Sigma} {}^{(sur)}J. \quad (4.17)$$

Recall that Σ is a small connected part of a 3D-surface which is transversal to the boundary $\phi(x^i) = 0$.

Consequently, the inhomogeneous boundary condition reads

$$[H] \wedge \mathcal{D}(\varphi) = {}^{(sur)}J. \quad (4.18)$$

Using (3.25) we can rewrite this equation in terms of the surface 2-form L ,

$$\int_{\Sigma} ([H] - L) \wedge \mathcal{D}(\varphi) = 0. \quad (4.19)$$

This equation must hold for an arbitrary choice of the integration domain Σ , thus the second boundary condition is:

$$([H] - L) \wedge \mathcal{D}(\varphi) = 0. \quad (4.20)$$

When the equation $\varphi(x^i) = 0$ is substituted here, we arrive at the non-singular condition

$$([H] - L) \wedge d\varphi = 0. \quad (4.21)$$

5 Boundary conditions from the integral Maxwell equations

In this section, we derive the boundary conditions from the integral Maxwell equations. These equations allow to consider non-differentiable and even non-continuous fields. Notice a simple fact: When we integrate the 3-forms dF , $d\mathcal{H}$ and \mathcal{J} , the domains of integration must be chosen to be a part of a 3-dimensional surfaces embedded in a 4-dimensional spacetime manifold. Similarly, the integrals of the 2-forms F and \mathcal{H} can be considered only over 2-dimensional domains. In the literature, see for instance [18], we can find integrals of the corresponding tensorial quantities taken over 4-dimensional regions. Such a procedure is certainly not invariant. If we use differential forms instead of tensors, we can determine the appropriate domains of integration.

5.1 Field strength jump condition

In the integral form, the homogeneous Maxwell equation $dF = 0$ is written as

$$\int_C F = 0, \quad (5.1)$$

where C is a 2-dimensional boundary of some connected 3-dimensional domain Σ , i.e., $C = \partial\Sigma$. Consequently, $\partial C = 0$.

Let Σ denote a part of a 3-dimensional surface (Fig 1.) that is transversal to the boundary $\varphi(x^i) = 0$ and bounded by the 3-surfaces $\varphi(x^i) = +\epsilon$ and $\varphi(x^i) = -\epsilon$. Geometrically it means that the boundary $\varphi(x^i) = 0$ divides Σ into two connected disjoint pieces Σ_+ and Σ_- . The 2-form F is assumed to be smooth in the domains Σ_+ and Σ_- . In the whole domain Σ , it is discontinuous, but finite (bounded).

Denote the parts of the boundary $\partial\Sigma$ lying on the surfaces $\varphi(x^i) = \pm\epsilon$ by $C_{+\epsilon}$ and $C_{-\epsilon}$, respectively. Let \tilde{C} denotes the remaining part of $\partial\Sigma$, see Fig.

1. Due to (5.1), we have

$$\int_{C_{+\epsilon}} {}^{(+)}F + \int_{C_{-\epsilon}} {}^{(-)}F + \int_{\tilde{C}} F = 0. \quad (5.2)$$

When the limit $\epsilon \rightarrow 0$ is taken, the third integral goes to zero, since the domain of integration \tilde{C} approaches zero. The domains $C_{+\epsilon}$ and $C_{-\epsilon}$ approach the same domain C_0 lying in the 3-dimensional surface $\varphi = 0$. Due to the opposite orientations of the domains $C_{+\epsilon}$ and $C_{-\epsilon}$, we are left with

$$\int_{C_0} [F] = 0. \quad (5.3)$$

Since C_0 is an arbitrary domain embedded in the surface with a constant value of the function φ , the recent equation yields

$$[F] \wedge d\varphi = 0. \quad (5.4)$$

5.2 Excitation jump condition

The inhomogeneous Maxwell equation $d\mathcal{H} = \mathcal{J}$ can be written in the integral form as

$$\int_{\partial\Sigma} \mathcal{H} = \int_{\Sigma} \mathcal{J}. \quad (5.5)$$

Here Σ is an arbitrary 3-dimensional domain bounded by the closed 2-dimensional surface $\partial\Sigma$. Using the same domain as above, we have

$$\int_{\partial\Sigma} \mathcal{H} = \int_{C_{+\epsilon}} {}^{(+)}H + \int_{C_{-\epsilon}} {}^{(-)}\mathcal{H} + \int_{\tilde{C}} \mathcal{H}. \quad (5.6)$$

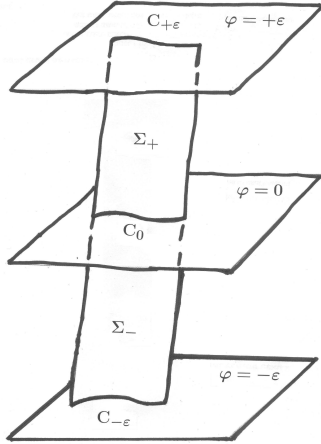


Fig. 1. Here Σ is a part of a 3-dimensional surface, which is transversal to the boundary $\varphi(x^i) = 0$ and bounded by the 3-surfaces $\varphi(x^i) = +\epsilon$ and $\varphi(x^i) = -\epsilon$. The boundary $\varphi(x^i) = 0$ divides Σ into two connected disjoint pieces Σ_{+} and Σ_{-} . The parts of the boundary $\partial\Sigma$ lying on the surfaces $\varphi(x^i) = \pm\epsilon$ are denoted by $C_{+\epsilon}$ and $C_{-\epsilon}$ correspondingly. \tilde{C} denotes the remaining part of $\partial\Sigma$.

When the limit $\epsilon \rightarrow 0$ is considered, the third integral trends to zero, while the domains C_+ and C_- go to the same limit C_0 . Hence, we are left with

$$\int_{\partial\Sigma} \mathcal{H} = \int_{C_0} [\mathcal{H}]. \quad (5.7)$$

The total current containing in the domain Σ consists from the bulk and surface current

$$\int_{\Sigma} \mathcal{J} = \int_{\Sigma} \left({}^{(\text{bulk})} \mathcal{J} + {}^{(\text{surf})} \mathcal{J} \right). \quad (5.8)$$

In the limit $\epsilon \rightarrow 0$, the bulk current vanishes, thus the total charge is

$$\int_{\Sigma} \mathcal{J} = \int_{\Sigma} L \wedge \mathcal{D}\varphi = \int_{C_0} L. \quad (5.9)$$

Equating (5.7) and (5.9), we obtain

$$\int_{C_0} \left([H] - L \right) = 0. \quad (5.10)$$

This integral equation is of the same type as (5.3). Due to the arbitrariness of the domains C_0 , it yields the boundary condition

$$\left([H] - L \right) \wedge d\varphi = 0. \quad (5.11)$$

6 Analysis of boundary conditions

In this section, we analyze the boundary conditions derived above:

$$\boxed{[F] \wedge d\varphi = 0, \quad \left([H] - L \right) \wedge d\varphi = 0.} \quad (6.1)$$

6.1 Boundary conditions in tensorial form

Although the differential form representation is very compact, the tensorial representation is also useful, particularly for a comparison with the literature. Let a part of a boundary be embedded in a domain with the coordinates x^i , $i = 0, 1, 2, 3$. In these coordinates, the differential of the boundary function is expressed as $d\varphi = \varphi_{,k} dx^k$. The coordinate representations of the fields are given in (2.3). The first equation of (6.1) can be rewritten as

$$[F_{ij}] \varphi_{,k} dx^i \wedge dx^j \wedge dx^k = 0. \quad (6.2)$$

We can even remove the basis 1-forms, but we must preserve the anti-symmetrization in the indices. This can be done by using the permutation pseudo-tensor. So we write the first boundary condition as

$$\varepsilon^{ijkl} [F_{ij}] \varphi_{,k} = 0, \quad (6.3)$$

The second equation of (6.1) is of the same structure as the first one. We only need a coordinate representation of the 2-form L . It is given in (3.27). Hence, the second equation becomes

$$\left([H^{ij}] - L^{ij}\right)\varphi_{,m}\varepsilon_{ijkl}dx^k \wedge dx^l \wedge dx^m = 0. \quad (6.4)$$

The basis forms can also be removed here, but the anti-symmetrization must be preserved, i.e., an additional permutation pseudo-tensor must be used. Thus, we arrive at the tensorial representation of the boundary conditions,

$$\boxed{\varepsilon^{ijkl}[F_{jk}]\varphi_{,l} = 0, \quad \left([H^{ij}] - L^{ij}\right)\varphi_{,j} = 0.} \quad (6.5)$$

6.2 Spacetime decomposition

Let us fix the coordinate system and decompose the 4-dimensional tensorial fields into 3-dimensional vector fields. The decomposition of the fields is given in (2.6–2.9). Denote the components of the 4D gradient $\varphi_{,i}$ by

$$\varphi_{,0} = n_0, \quad \varphi_{,\alpha} = n_\alpha (= \mathbf{n}). \quad (6.6)$$

Recall that the Greek indices run over 1, 2, and 3.

Homogeneous conditions: For $i = 0$, we rewrite (6.5) as

$$\varepsilon^{\alpha\beta\gamma}[F_{\alpha\beta}]\varphi_{,\gamma} = 0, \quad (6.7)$$

where the standard identification $\varepsilon^{0\alpha\beta\gamma} = \varepsilon^{\alpha\beta\gamma}$ is used. Using (2.8), we have

$$[B^\alpha]\varphi_{,\alpha} = 0. \quad (6.8)$$

In ordinary vectorial notation, it can be rewritten as

$$[\mathbf{B}] \cdot \mathbf{n} = 0. \quad (6.9)$$

Recall that the dot-product used here is independent of the metric. It means a contraction of a vector with a covector, rather than a scalar product of two vectors. For a spatial value of the index i , say $i = \alpha$, we rewrite (6.5) as

$$\varepsilon^{\alpha 0 \beta \gamma}[F_{0\beta}]\varphi_{,\gamma} + \varepsilon^{\alpha \beta 0 \gamma}[F_{\beta 0}]\varphi_{,\gamma} + \varepsilon^{\alpha \beta \gamma 0}[F_{\beta \gamma}]\varphi_{,0} = 0. \quad (6.10)$$

Using (2.6), we have

$$\varepsilon^{\alpha \beta \gamma}[E_\beta]n_\gamma - [B^\alpha]n_0 = 0. \quad (6.11)$$

In vectorial notation, it is rewritten as

$$[\mathbf{E}] \times \mathbf{n} - [\mathbf{B}]n_0 = 0. \quad (6.12)$$

Inhomogeneous conditions: The structure of the second equation of (6.5) is similar to the first one. Thus, for $i = 0$, we have

$$[H^{0\alpha}] \varphi_{,\alpha} = L^{0\alpha} \varphi_{,\alpha}, \quad (6.13)$$

or, equivalently,

$$[D^\alpha] n_\alpha = -L^{0\alpha} n_\alpha. \quad (6.14)$$

For the spatial value of the index, say $i = \alpha$, we have

$$\left([H^{\alpha\beta}] - L^{\alpha\beta}\right) \varphi_{,\beta} + \left([H^{\alpha 0}] - L^{\alpha 0}\right) \varphi_{,0} = 0. \quad (6.15)$$

In terms of the 3D vectors, it is rewritten as

$$\varepsilon^{\alpha\beta\gamma} [H_\gamma] n_\beta + [D^\alpha] n_0 = L^{\alpha\beta} n_\beta + L^{\alpha 0} n_0. \quad (6.16)$$

Comparing with the inhomogeneous static boundary conditions (3.7), we rewrite (6.14) and (6.16) in vectorial form as

$$[\mathbf{D}] \cdot \mathbf{n} = \sigma, \quad [\mathbf{H}] \times \mathbf{n} + [\mathbf{D}] n_0 = \mathbf{K}. \quad (6.17)$$

Here the surface charge scalar and the surface current vector are defined as

$$\sigma = -L^{0\alpha} n_\alpha, \quad K^\alpha = -L^{\alpha\beta} n_\beta - L^{\alpha 0} n_0. \quad (6.18)$$

6.3 Independent conditions

The number of independent boundary conditions in electromagnetism is discussed in literature, see [20],[21] and the references given therein. In the differential form notations, the boundary conditions are written as

$$C_1 = [F] \wedge \mathcal{D}(\varphi) = 0, \quad C_2 = ([H] - L) \wedge \mathcal{D}(\varphi) = 0. \quad (6.19)$$

We have here two 3-forms C_1 and C_2 which are equal to zero, i.e., there are eight conditions linear in the field jumps. These conditions, however, are not independent. Indeed, one can easily observe two linear constraints

$$C_1 \wedge d\varphi = C_2 \wedge d\varphi = 0. \quad (6.20)$$

Thus, in the covariant differential form description, only six linear independent conditions remain.

In the vectorial representation, we have a system of eight equations

$$\begin{aligned} [\mathbf{B}] \cdot \mathbf{n} &= 0, & [\mathbf{D}] \cdot \mathbf{n} &= \sigma, \\ [\mathbf{E}] \times \mathbf{n} - [\mathbf{B}] n_0 &= 0, & [\mathbf{H}] \times \mathbf{n} + [\mathbf{D}] n_0 &= \mathbf{K}. \end{aligned} \quad (6.21)$$

Observe here two different possibilities:

Non-static boundary: For a moving boundary with $n_0 = \partial\varphi/\partial x^0 \neq 0$, two scalar equations in the first line of (6.21) are the results of the vector equations

given in the second line. To see these relations, it is enough to apply the dot-multiplication of the vector equations with the vector \mathbf{n} and use the charge conservation law. Thus, we are left with only six independent conditions:

$$[\mathbf{E}] \times \mathbf{n} - [\mathbf{B}]n_0 = 0, \quad [\mathbf{H}] \times \mathbf{n} + [\mathbf{D}]n_0 = \mathbf{K}. \quad (6.22)$$

This is in an agreement with the covariant consideration given above.

Static boundary. In this case when φ is independent of the x^0 coordinate, the scalar $n_0 = 0$, thus we are left with eight boundary conditions

$$[\mathbf{B}] \cdot \mathbf{n} = 0, \quad [\mathbf{D}] \cdot \mathbf{n} = \sigma, \quad (6.23)$$

$$[\mathbf{E}] \times \mathbf{n} = 0, \quad [\mathbf{H}] \times \mathbf{n} = \mathbf{K}. \quad (6.24)$$

Note that we did not use any constitutive relation so far. In this situation, four vector fields are completely independent, hence also the equations (6.23-6.24) seem to be independent. In fact, this is not the case. Indeed, in (6.23) we have two independent scalar conditions. As for the vector equations given in (6.24), not all of them are independent. Denoting $\mathbf{A}_1 = [\mathbf{E}] \times \mathbf{n}$, we can easily observe a linear relation $\mathbf{A}_1 \cdot \mathbf{n} = 0$. Similarly, for the second vector $\mathbf{A}_2 = [\mathbf{H}] \times \mathbf{n} - \mathbf{K}$, we have an identity $\mathbf{A}_2 \cdot \mathbf{n} = 0$, providing the surface current being tangential to the surface $\mathbf{K} \cdot \mathbf{n} = 0$. Thus, we have the same six boundary conditions as in the non-static case.

7 Energy transfer

7.1 Metric-free energy-momentum current

In relativistic physics, the energy of a field together with its momentum and its stress tensor form a four-dimensional energy-momentum tensor T^j_i . Due to the GR paradigm, the conservation law for this tensor must be expressed by use of the covariant derivative $T^j_{i;j} = 0$. Although this equation is ordinarily used as the GR representation of the conservation law, it does not provide an invariant integral conservation law. In the differential form representation, the energy-momentum current Σ_i is considered instead of the tensor T^j_i . The relation between the tensor T^j_i and the form Σ_i is provided by use of the volume element and does not require the metric structure [28]. Σ_i is a covector valued 3-form that obeys the following conservation law:

$$d\Sigma_i = \mathcal{F}_i, \quad (7.1)$$

where \mathcal{F}_i is a covector valued 4-form of the force density. It should be noted that the equation (7.1) is not invariant under general coordinate transformations, and the integral formulation is not applicable also in this formulation.

The same problem appears in the electromagnetism of media. Different definitions of the electromagnetic energy-momentum tensor are known to be

applicable in different circumstances, but it is not clear yet from both a theoretical [29] and an experimental [30] point of view, which of them must be taken as the fundamental one. For the purpose of the analysis of the boundary conditions, we will apply the definition which can be traced back to Minkowski. A corresponding tensor (form) is derived from the electromagnetic Lagrangian by the standard Noether-Hilbert procedure. This fact distinguishes Minkowski's tensor from the other energy-momentum expressions.

In the premetric approach, the energy-momentum current of the electromagnetic field is expressed as

$$\Sigma_i = \frac{1}{2} [(e_i \rfloor F) \wedge H - (e_i \rfloor H) \wedge F] , \quad (7.2)$$

where the interior multiplication operator \rfloor is applied.

When the exterior derivative of this expression is evaluated, the corresponding force density $\mathcal{F}_i = d\Sigma_i$ includes the Lorentz force density $(e_i \rfloor F) \wedge J$ plus an additional force density. This additional force can be expressed via the Lie derivatives of the fields F and H taken relative to the coordinate vectors [5]. In the particularly interesting case of a linear constitutive relation $H^{ij} = (1/2)\chi^{ijkl}F_{kl}$, the additional force is expressed by the first-order derivatives of the constitutive tensor χ^{ijkl} . It should be noted that this additional force is not only a mathematical artifact, but is observed in experiments [30].

7.2 Energy-momentum boundary conditions

To describe the distribution of the energy-momentum current in the case of two media separated by a hypersurface $\varphi(x^i)$, we substitute the definitions (4.1, 4.11) of the fields into (7.2). Using the identities

$$u(\varphi)u(-\varphi) = 0, \quad [u(\varphi)]^2 = u(\varphi), \quad (7.3)$$

we derive the decomposition of the energy-momentum current as the sum of two independent currents related to the two different media

$$\Sigma_i = {}^{(-)}\Sigma_i u(\varphi) + {}^{(+)}\Sigma_i u(-\varphi). \quad (7.4)$$

Here

$${}^{(-)}\Sigma_i = \frac{1}{2} \left[(e_i \rfloor {}^{(-)}F) \wedge {}^{(-)}H - (e_i \rfloor {}^{(-)}H) \wedge {}^{(-)}F \right], \quad (7.5)$$

and similarly for ${}^{(+)}\Sigma_i$. Consequently,

$$\begin{aligned} d\Sigma_i &= d\left({}^{(-)}\Sigma_i\right) u(\varphi) + d\left({}^{(+)}\Sigma_i\right) u(-\varphi) + \left({}^{(-)}\Sigma_i - {}^{(+)}\Sigma_i\right) \wedge \mathcal{D}(\varphi) \\ &= {}^{(-)}\mathcal{F}_i u(\varphi) + {}^{(+)}\mathcal{F}_i u(-\varphi) + \left({}^{(-)}\Sigma_i - {}^{(+)}\Sigma_i\right) \wedge \mathcal{D}(\varphi). \end{aligned} \quad (7.6)$$

Thus, the jump of the energy-momentum current emerges as an additional force acting on the boundary hypersurface

$${}^{(add)}\mathcal{F}_i = \left[\Sigma_i\right] \wedge \mathcal{D}(\varphi) = \left[\Sigma_i\right] \wedge \delta(\varphi)d\varphi. \quad (7.7)$$

This force is localized on the boundary hypersurface and is proportional to the 4D-gradient $d\varphi$.

7.3 Tensorial description of the energy-momentum jump

Recall that the considerations above do not require any geometrical structure on the manifold. In order to deal with the ordinary notion of the energy-momentum tensor, we need a volume element to be defined on the manifold. Conventionally, this quantity is defined by use of the metric tensor $vol \sim \sqrt{-g}$. Instead, we will consider the volume element vol as a fundamental structure without any relation to the metric. Notice that vol must be a smooth non-vanishing twisted 4-form with the support on the whole manifold.

When a volume element is defined, the covector-valued 4-form of the force density is expressed as

$$\mathcal{F}_i = f_i vol. \quad (7.8)$$

Here f_i is the ordinary (non-twisted) covector.

The twisted 3-form of energy-momentum current is expressed by use of the contraction (inner product) operator $e_i \rfloor$.

$$\Sigma_i = -T_i^j e_j \rfloor vol. \quad (7.9)$$

For a coordinate frame $e_i = \partial/\partial x^i$, it is the standard partial derivative operator. Recall the known identity $e_i \rfloor dx^j = \delta_i^j$. Substituting (7.8, 7.9) into (7.7), we obtain

$$^{(add)} f_i vol = -[T_i^j] \varphi_m \delta(\varphi) (e_j \rfloor vol \wedge dx^m) = [T_i^j] \varphi_j \delta(\varphi) vol. \quad (7.10)$$

Here, we used an identity $(e_j \rfloor vol) \wedge dx^m = \delta_j^m vol$. From (7.10), we have the energy-momentum jump conditions in the tensorial representation

$$^{(add)} f_i = [T_i^j] \varphi_j \delta(\varphi). \quad (7.11)$$

Up to a normalization factor, this result coincides with the one given in [18]. The space-time decomposed form of this equation can be also found in [18].

The results of this section are independent of the metric and of the volume element structures, it is clear from the differential form representation (7.7). Although the volume element was used for the definitions of the vector f_i and of the tensor T_i^j , the relation (7.11) is independent of the specific choice of the volume form. Moreover, every two volume elements differ only by a scalar factor which cancels on both sides of (7.11).

8 Example: Moving boundary in an isotropic medium

As a simple application of the covariant jump conditions derived above, we consider the following question:

Let on both sides of the hypersurface $\varphi = 0$ be the same medium. Is it possible to have a jump of the field even in the absence of surface charges and currents?

We will consider a moving surface, i.e. one described in a chosen system of coordinates by the equation $\varphi(x^i) = 0$ with the derivatives

$$\varphi_{,0} = n_0, \quad \varphi_{,\alpha} = n_\alpha. \quad (8.1)$$

As it was described above, the function φ is defined up to the action of an arbitrary monotonic function $f(\varphi)$. In particular, we can multiply φ by an arbitrary constant. Thus, we can assume the spatial part n_i to be dimensionless. Consequently, n_0 has the dimension of velocity. In contrast to the motion of a point described by a velocity vector, we need only a scalar in order to describe the motion of a hypersurface. As an example, a flat uniformly moving hyperplane will be described by a linear equation

$$\varphi = n_\alpha x^\alpha - n_0 t = 0 \quad (8.2)$$

with a constant 4D-covector $n_i = (n_0, n_\alpha)$ and a constant scalar speed n_0 .

The notion of an isotropic medium notion can only be defined in the 3-dimensional sense. It is described by two parameters: the permittivity ε and the permeability μ that appear in the constitutive relations

$$D^\alpha = \varepsilon \delta^{\alpha\beta} E_\beta, \quad \mathcal{H}_\alpha = \frac{1}{\mu} \delta_{\alpha\beta} B^\beta. \quad (8.3)$$

We do not need to restrict ourselves to constant parameters. Hence ε and μ can depend on the position and the time coordinates. Since the definition of the isotropic medium involves the 3-dimensional metric $\delta_{\alpha\beta} = \text{diag}(1, 1, 1)$, we can use it for lowering and raising indices. So we return to the standard vectorial notation and write the constitutive relation in the ordinary textbook form

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}. \quad (8.4)$$

Let us first check what we can get from the static boundary conditions. Substituting (8.4) into (3.6, 3.7), we have (recall the vector notation $\mathbf{n} = n_\alpha$)

$$\mu \mathbf{n} \cdot [\mathbf{H}] = 0, \quad [\mathbf{E}] \times \mathbf{n} = 0, \quad (8.5)$$

$$\varepsilon \mathbf{n} \cdot [\mathbf{E}] = 0, \quad [\mathbf{H}] \times \mathbf{n} = 0. \quad (8.6)$$

Using the vector triple product, we get

$$\mathbf{n} \times ([\mathbf{E}] \times \mathbf{n}) = [\mathbf{E}] n^2 - \mathbf{n} (\mathbf{n} \cdot [\mathbf{E}]) = [\mathbf{E}] n^2. \quad (8.7)$$

Due to (8.5), we have $[\mathbf{E}] n^2 = 0$. Since the Euclidean square $\mathbf{n}^2 \neq 0$, we obtain that, in the isotropic medium, the jump of the electromagnetic field on a static surface is forbidden.

Let us now substitute the constitutive relations (8.4) into the dynamical jump conditions (6.21). Due to the parametrization (8.1), we obtain

$$\begin{aligned}\mu[\mathbf{H}] \cdot \mathbf{n} &= 0, & \varepsilon[\mathbf{E}] \cdot \mathbf{n} &= 0, \\ [\mathbf{E}] \times \mathbf{n} - \mu[\mathbf{H}]n_0 &= 0, & [\mathbf{H}] \times \mathbf{n} + \varepsilon[\mathbf{E}]n_0 &= 0.\end{aligned}\quad (8.8)$$

Now we have

$$\mathbf{n} \times ([\mathbf{E}] \times \mathbf{n}) = [\mathbf{E}]\mathbf{n}^2, \quad (8.9)$$

which is not equal to zero. Due to (8.8), this expression is evaluated as

$$\mathbf{n} \times ([\mathbf{E}] \times \mathbf{n}) = \mu n_0 \mathbf{n} \times [\mathbf{H}] = \mu \varepsilon n_0^2 [\mathbf{E}]. \quad (8.10)$$

Thus, we obtain a jump condition

$$(\mathbf{n}^2 - \mu \varepsilon n_0^2) [\mathbf{E}] = 0. \quad (8.11)$$

Consequently, a non-zero jump of the field \mathbf{E} is admissible on a surface $\varphi(x^i) = 0$ whose gradient $\varphi_i = (n_0, \mathbf{n})$ satisfies the relation

$$\mathbf{n}^2 - \mu \varepsilon n_0^2 = 0. \quad (8.12)$$

The expression on the left-hand-side here can be viewed as a Lorentzian type norm of the covector φ_i . The constant of the speed of light in vacuum in this expression is substituted by the speed of light in the medium $v = n_0/|\mathbf{n}|$

$$v = \frac{1}{\sqrt{\mu \varepsilon}}. \quad (8.13)$$

Thus the 4-dimensional metric emerges in a priori metric-free background due to the special isotropic constitutive relations. This metric is Lorentzian if the product $\mu \varepsilon$ positive. This condition holds for the ordinary materials with $\mu > 0, \varepsilon > 0$ as well for the recently generated metamaterials with the parameters $\mu < 0, \varepsilon < 0$. When the parameters μ, ε are of different signs, the metric is Euclidean. The non-zero jump of the fields is usually described as the shock waves. Note that we derived such type of behavior from the jump conditions alone, without using the field equations directly and without assuming any wave-type ansatz.

9 Results and discussion

We derived the covariant boundary conditions in electrodynamics. The resulting formulas as well as the derivation procedures do not require the metric structure. Thus, these results are valid on an arbitrary differential manifold. A key point of our consideration is a proper metric-free redefinition of the electromagnetic fields, the electric current and the energy-momentum current. These quantities are represented by singular differential forms and require Dirac's delta-form for their definitions. The metric-free covariant boundary conditions must have a

wide area of applications from GR to the electromagnetism of media. Recently, Kurz and Heumann [31] applied the premetric covariant boundary conditions numerically to the classical Wilson experiment.

Our derivation was based on the boundary between two different media. However, due to their metric-free form, the results must hold for an arbitrary hypersurface. In particular, in the source-free form, they are applicable to the wave front surface. A simple isotropic example of such a consideration is presented above.

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References

- [1] A. Sommerfeld, *Electrodynamics* (New York: Academic Press, 1964).
- [2] W. Pauli, *Theory of Relativity* (Pergamon, Oxford, 1958).
- [3] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, Oxford, 1984).
- [4] J. D. Jackson, *Classical Electrodynamics* (3ed., Wiley, New York, 1999).
- [5] F. W. Hehl and Yu. N. Obukhov, *Foundations of Classical Electrodynamics: Charge, Flux, and Metric* (Birkhäuser: Boston, MA, 2003).
- [6] C. Lammerzahl, Lect. Notes Phys. **702**, 349-384 (2006).
- [7] V. A. Kostelecky, M. Mewes, Phys. Rev. **D66**, 056005 (2002).
- [8] F. W. Hehl, Ann. Phys. (Berlin) **17**, 691-704 (2008).
- [9] T. Damour, Ann. Phys. (Berlin) **17**, 619-630 (2008).
- [10] F. Kottler, Sitzungsber. Akad. Wien IIa **131**, 119 (1922).
- [11] E. Cartan, Ann. Ec. Norm. Super. **41**, 1, (1924).
- [12] D. Van Dantzig, Proc. Cambridge Phil. Soc. **30**, 421 (1934); Proc. Acad. Amsterdam **37**, 521, 526, 644, 825 (1934).
- [13] E. J. Post, *Formal Structure of Electromagnetics – General Covariance and Electromagnetics* (North Holland: Amsterdam, 1962).
- [14] V. Perlick, Gen. Relativ. Gravitation **38**, 365-380 (2006).

- [15] Y. Itin, J. Phys. A **42**, 475402 (2009).
- [16] Y. Itin, J. Phys. A **40**, F737 (2007).
- [17] Y. Itin, Physics Letters A **374**, 1113-1116 (2010).
- [18] R. C. Costen, “Four-dimensional Derivation of the Electrodynamic Jump conditions, Traction, and Power Transfer at a Moving Boundary”, NASA Technical Notes, D-2618 (1965).
- [19] D. Rawson-Harris, International Journal of Theoretical Physics **6**, 339-346 (1972).
- [20] C. Yeh, Phys. Rev. E **48**, 14261427 (1993).
- [21] H. Gur, PIER **23**, 107-136 (1999).
- [22] I.V. Lindell and B. Jancewicz, Eur. J. Phys. **21**, 83-89 (2000).
- [23] P. Hillion, Phys. Rev. A **41**, 3449 (1990).
- [24] Y. Itin and F. W. Hehl, Annals Phys. (N.Y.) **312**, 60 (2004).
- [25] F. W. Hehl and Yu. N. Obukhov, Gen. Rel. Grav. **37**, 733 (2005).
- [26] G. de Rham, *Differentiable Manifolds*, (Springer-Verlag, 1980).
- [27] Y. Itin, “Low-dimensional electric charges. Covariant description,” arXiv:1006.1199 [math-ph].
- [28] Y. Itin, J. Phys. A **36**, 8867-8884 (2003).
- [29] T. Ramos, G. F. Rubilar, Y. N. Obukhov, Phys. Lett. **A375**, 1703-1709 (2011).
- [30] I. H. Brevik, Phys. Rept. **52**, 133-201 (1979).
- [31] S. Kurz and H. Heumann, ”Transmission conditions in pre-metric electrodynamics”, ETH Zürich Research Report No. 2010-28 (2010).